

Gauge stability of 3+1 formulations of General Relativity

A.M Khokhlov* and I.D. Novikov^{†‡§¶}

February 7, 2008

Abstract

We present a general approach to the analysis of gauge stability of 3+1 formulations of General Relativity (GR). Evolution of coordinate perturbations and the corresponding perturbations of lapse and shift can be described by a system of eight quasi-linear partial differential equations. Stability with respect to gauge perturbations depends on a choice of gauge and a background metric, but it does not depend on a particular form of a 3+1 system if its constrained solutions are equivalent to those of the Einstein equations. Stability of a number of known gauges is investigated in the limit of short-wavelength perturbations. All fixed gauges except a synchronous gauge are found to be ill-posed. A maximal slicing gauge and its parabolic extension are shown to be ill-posed as well. A necessary condition is derived for well-posedness of metric-dependent algebraic gauges. Well-posed metric-dependent gauges are found, however, to be generally unstable. Both instability and ill-posedness are associated with perturbations of physical accelerations of reference frames.

*Laboratory for Computational Physics, Code 6404, Naval Research Laboratory, Washington, DC 20375

[†]Theoretical Astrophysics Center, Juliane Maries vej 30, DK-2100 Copenhagen, Denmark

[‡]Copenhagen University Observatory, Juliane Maries vej 30, DK-2100 Copenhagen, Denmark

[§]Astro Space center of the P.N. Lebedev Physical Institute, Profsoyuznaja 84/32, Moscow 118710, Russia

[¶]NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

1 Introduction

Physical analysis of many problems of general relativity (GR) requires a solution of a full set of the non-linear Einstein equations. Due to a complexity of these equations, in most cases this can be done only numerically. This paper is concerned with gauge stability of 3+1 approaches to numerical integration of the Einstein equations. 3+1 means here any approach in which the equations are split on constraint and evolution parts. Constraints are satisfied on an initial three-dimensional space-like hypersurface. Initial data is then evolved in time by solving a Cauchy problem.

The original ADM 3+1 formulation uses a three-dimensional metric γ_{ij} and an extrinsic curvature K_{ij} as unknown functions [1]. In other 3+1 formulations, such as hyperbolic, conformal formulations and their combinations, the ADM equations are extended by introducing additional variables such as spatial derivatives of γ_{ij} , traces of γ_{ij} and K_{ij} , conformal factors, by forming new combinations of these variables, or by modifying equations with the help of constraints [2-18]. These modifications change the nature of the equations so that they can become more stable, and the integration can be prolonged [19-23]. Nonetheless, calculations (of black hole collisions, in particular) often suffer from instabilities. A stable, accurate, long-term integration of the Einstein equations remains as an outstanding problem of numerical GR.

Difficulties with time-integration can be numerical and analytical. Numerical difficulties arise when an unstable finite-difference scheme is used to integrate a particular 3+1 set of equations. Constructing a stable numerical scheme for a chosen 3+1 set of equations should not be a problem if the equations are well-posed and stable. Numerical stability will not be discussed in this paper (see, e.g., [16, 25, 26] for numerical aspects). We believe that analytical difficulties arising from the nature of the Einstein equations themselves present a much more serious and still unsolved problem.

First of all, a 3+1 set of GR equations must be well-posed [27, 28]. It is impossible to guarantee for an ill-posed system a convergence of solutions when numerical resolution is increased. It is known, for example, that a harmonic gauge leads to a symmetric hyperbolic, well-posed system [29, 30]. Some gauges lead to ill-posed systems (see below). The property of well-posedness is local and time-dependent. Using a gauge that maintain well-posedness everywhere and at all times is crucial for a long-term stable integration.

Maintaining well-posedness, however, is not enough. A well-posed non-linear system may still have rapidly growing, diverging, or singular solutions so that numerical integration will become problematic. Here we call this an analytical instability. Analytical instabilities can be separated on three types. First, the instability may be related to a physical nature of space-time. For example, one can encounter a true singularity where curvature invariants become infinite. Second, the instability may be related to violation of constraints during time-integration (constraint instability). These may be the energy and momentum constraints, as well as additional constraints arising from introduction of additional variables into a system. If constraints are satisfied initially, they are automatically satisfied at later times. However, a Cauchy problem possesses a much broader class of solutions than constraint-satisfying solutions. Consequently, a small initial perturbation may lead to a rapid deviation from a constrained solution during integration. The third type is the gauge

instability. In a 3+1 approach, a coordinate system is “constructed” during time integration according to a pre-determined choice of gauge and initial conditions. Initially small coordinate perturbations may lead to a divergence of coordinate systems as time progresses. For a long-term integration to be successful, a gauge must be both well-posed and stable.

Little is known about stability properties of 3+1 systems of GR equations in general. The analysis is complicated because many different modes are present simultaneously, stability is determined by complex non-linear terms in the Einstein equations, and it depends on a particular solution and on location in spacetime. Investigation of well-posedness is somewhat easier since it requires analyzing only a principal part of the system. Several classes of hyperbolic, well-posed 3+1 formulations have been constructed (e.g., [5, 10, 13, 18, 19]), but numerical experiments show that these systems are often unstable. There are indications that, at least partially, the instability is related to constraint violating modes (e.g., [13, 19]).

It is long known that a synchronous gauge is prone to the formation of coordinate singularities (or caustics) [31, 32]. Consequences of this for a numerical integration were discussed most recently in [33]. An attempt to characterize the development of coordinate pathologies (shocks) in hyperbolic 3+1 formulations has been made in [34]. Perturbations of coordinates have been separated, in linear approximation, and studied for a synchronous gauge (see Problem 3 in Paragraph 95 of [31] and also [33]). To our knowledge, instability of gauges other than a synchronous gauge have not been analyzed.

In this paper we develop a general approach to the analysis of gauge instabilities. In Section 2 we derive a system of eight quasi-linear partial differential equations that describe coordinate perturbations for arbitrary gauges and in arbitrary 3+1 system. We illustrate our approach by investigating stability of various gauges in Section 3. A physical meaning of gauge instabilities is discussed in Section 4. Our conclusions are given in Section 5.

2 Equations of gauge perturbations

Consider a space-time described in a certain coordinate system x^a by a four-dimensional metric g_{ab} . In what follows we use letters $a - h$ to denote four-dimensional indices 0, 1, 2, 3, and letters $i - m$ to denote three-dimensional spatial indices 1, 2, 3; time $t = x^0$. We write an interval as [35]

$$ds^2 = g_{ab}dx^a dx^b = -(\alpha^2 - \beta_i \beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j \quad (2.1)$$

where α is a lapse, β_i is a shift, $\beta^i = \gamma^{ij}\beta_j$, $\gamma^{im}\gamma_{mk} = \delta^i_k$, γ_{ij} is a metric on a three-dimensional space-like hypersurface, and the 4-metric is

$$g_{ab} = \begin{pmatrix} -(\alpha^2 + \beta_i \beta^i) & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}, \quad g_{ab}g^{bc} = \delta_a^c. \quad (2.2)$$

We want to separate, in a linear approximation, the behavior of coordinates from physical evolution of a space-time. Under an infinitesimal coordinate (gauge) transformation $\psi^a(x^0, x^1, x^2, x^3)$ of a four-dimensional coordinate system,

$$x^a \rightarrow x^a + \psi^a, \quad (2.3)$$

where ψ^a are sufficiently smooth infinitesimal functions, the metric transforms as [31]

$$g_{ab} \rightarrow g_{ab} + \delta g_{ab} , \quad (2.4)$$

where

$$\delta g_{ab} = -(\nabla_b \psi_a + \nabla_a \psi_b) \quad (2.5)$$

are the deviations of the metric, and ∇ denotes a covariant derivative.

For a metric (2.1), the deviations δg_{ab} can also be related to corresponding deviations of lapse, shift, and three-dimensional metric,

$$U \equiv \delta \alpha , \quad V_i \equiv \delta \beta_i , \quad W_{ij} \equiv \delta \gamma_{ij} \quad (2.6)$$

as

$$\begin{aligned} \delta g_{00} &= -2\alpha U + 2\gamma^{ij} \beta_i V_j - \gamma^{ik} \beta_i \beta_j W_{jk} \\ \delta g_{0i} &= V_i \\ \delta g_{ij} &= W_{ij} \end{aligned} \quad (2.7)$$

Substitution of (2.7) into (2.5) gives us U , V_i and W_{ij} in terms of ψ_a ,

$$U = \frac{1}{\alpha} \left(\frac{\beta^i \beta^j W_{ij}}{2} - \beta^i V_i - \frac{\partial \psi_0}{\partial t} + \Gamma_{00}^c \psi_c \right) , \quad (2.8)$$

$$V_i = - \left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_0}{\partial x^i} \right) + 2\Gamma_{0i}^c \psi_c , \quad (2.9)$$

$$W_{ij} = - \left(\frac{\partial \psi_j}{\partial x^i} + \frac{\partial \psi_i}{\partial x^j} \right) + 2\Gamma_{ij}^c \psi_c , \quad (2.10)$$

where Γ_{ab}^c are four-dimensional Cristoffel symbols. They can be expressed in terms of α , β_i , and γ_{ij} using (2.2). Equations (2.8) and (2.9) can also be rewritten as a system of four quasi-linear partial differential equations for ψ_a ,

$$\begin{aligned} \frac{\partial \psi_0}{\partial t} &= \alpha U - \beta^i V_i - \beta^i \beta^j \frac{\partial \psi_i}{\partial x^j} + (\Gamma_{00}^c + \beta^i \beta^j \Gamma_{ij}^c) \psi_c , \\ \frac{\partial \psi_i}{\partial t} &= -V_i - \frac{\partial \psi_0}{\partial x^i} + 2\Gamma_{0i}^c \psi_c . \end{aligned} \quad (2.11)$$

The deviations ψ^a , and the corresponding deviations U , V_i were arbitrary up to this point.

During the integration of a 3+1 system in time, certain conditions are imposed on α and β_i which specify the choice of gauge. In general, a gauge can be specified by a set of four partial differential equations with respect to lapse, shift, three-dimensional metric, and their partial derivatives,

$$F_a \left(x^b, \alpha, \frac{\partial \alpha}{\partial x^b}, \dots, \beta_i, \frac{\partial \beta_i}{\partial x^b}, \dots, \gamma_{ij}, \frac{\partial \gamma_{ij}}{\partial x^b}, \dots \right) = 0 , \quad (2.12)$$

where ... in (2.12) indicate higher order derivatives of α , β_i and γ_{ij} . Equations (2.12) must be considered a part of a 3+1 system which is integrated in time along with the rest of the equations for metric components, extrinsic curvature, etc. By varying (2.12) with respect to α , β_i and γ_{ij} and making use of (2.10) we obtain a system of four additional quasi-linear partial differential equations relating U and V_i to coordinate perturbations ψ^a ,

$$\begin{aligned} & \frac{\partial F_a}{\partial \alpha} U + \frac{\partial F_a}{\partial \left(\frac{\partial \alpha}{\partial x^b}\right)} \frac{\partial U}{\partial x^b} + \frac{\partial F_a}{\partial \left(\frac{\partial^2 \alpha}{\partial x^b \partial x^c}\right)} \frac{\partial^2 U}{\partial x^b \partial x^c} + \dots + \\ & \frac{\partial F_a}{\partial \beta_i} V_i + \frac{\partial F_a}{\partial \left(\frac{\partial \beta_i}{\partial x^b}\right)} \frac{\partial V_i}{\partial x^b} + \frac{\partial F_a}{\partial \left(\frac{\partial^2 \beta_i}{\partial x^b \partial x^c}\right)} \frac{\partial^2 V_i}{\partial x^b \partial x^c} + \dots + \\ & 2 \frac{\partial F^a}{\partial \gamma_{ij}} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + 2 \frac{\partial F^a}{\partial \left(\frac{\partial \gamma_{ij}}{\partial x^b}\right)} \frac{\partial}{\partial x^b} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + \dots = 0 \end{aligned} \quad (2.13)$$

Together, (2.11) and (2.13) form a system of eight equations describing the evolution of coordinate perturbations and associated perturbations of lapse and shift in time. Perturbations of a three-dimensional metric which correspond to evolving coordinate perturbations are given by (2.10).

At this point, we succeeded in separating the behavior of gauge perturbations from other possible perturbations of the solutions of a 3+1 system. Gauge instabilities can be studied by investigating a set of eight quasi-linear partial differential equations (2.11) and (2.13) for coordinate perturbations ψ_a and associated perturbations of lapse and shift, U and V_i . During the derivation, we did not use any specific assumptions about a 3+1 system. The only assumption implicit to the derivation was the equivalence of constrained solutions of a 3+1 system to these of the Einstein equations. Therefore, (2.11) and (2.13) must be applicable to any such system. In a linear approximation, the behavior of gauge modes of perturbations depends only on a choice of gauge and on an unperturbed metric g_{ab} .

Since coefficients of (2.11), (2.13) are functions of g_{ab} and thus of x^a , exact solutions of (2.11), (2.13) can be found only in some special cases. A further simplification is possible in a high-frequency limit where we consider coordinate perturbations on much shorter scales compared to those of a base solution g_{ab} . Let us designate

$$\vec{z}(x^a) = \{\psi^a, U, V_i\}^T \quad (2.14)$$

a combined vector of unknowns entering our gauge perturbation equations. We will consider in this paper the gauge equations (2.13) which involve γ_{ij} and its first-order time derivatives (cases with higher order derivatives can be treated similarly). Then the resulting gauge perturbation equations will contain only first-order time derivatives of \vec{z} , but, of course, they can contain higher-order spatial derivatives of \vec{z} . We write (2.11), (2.13) symbolically as

$$\frac{\partial z_r}{\partial t} = {}^{(0)}\mathcal{M}_{rs} z_s + {}^{(1)}\mathcal{M}_{rs}^i \frac{\partial z_s}{\partial x^i} + {}^{(2)}\mathcal{M}_{rs}^{ij} \frac{\partial^2 z_s}{\partial x^i \partial x^j} + \dots, \quad (2.15)$$

where ${}^{(k)}\mathcal{M}$, $k = 0, 1, 2, \dots$ depend on an unperturbed solution. In the limit of short-wavelength perturbations, we look for solutions of (2.15) in the form

$$\vec{z} = \vec{\zeta}(t) \exp(-I q e_k x^k), \quad (2.16)$$

where $I = \sqrt{-1}$, $q_i = qe_i$ is a wavevector, e_i is a unit vector, and q is an absolute value of q_i . Substituting (2.16) into (2.15) gives an ordinary differential equation for $\vec{\zeta}$ (we retain only the time-dependence of $^{(k)}\mathcal{M}$),

$$\frac{d\vec{\zeta}}{dt} = \hat{M}(t) \vec{\zeta} , \quad (2.17)$$

with

$$M_{rs}(t, e_i, q) = {}^{(0)}\mathcal{M}_{rs}(t) + {}^{(1)}\mathcal{M}_{rs}^i(t) I e_i q + {}^{(2)}\mathcal{M}_{rs}^{ij}(t) e_i e_j q^2 + \dots \quad (2.18)$$

When the time-dependence of \hat{M} can be neglected, the solutions become

$$\vec{z}(x^a) \propto \exp(\omega_s t - I q e_k x^k) , \quad (2.19)$$

where wavenumbers $\omega_s(q, e_i)$ are determined by the dispersion relation

$$\det(\hat{M} - \omega \hat{N}) = 0 , \quad (2.20)$$

and \hat{N} is a unit matrix. If a timescale of the variation of coefficients in \hat{M} becomes comparable to $|\omega|^{-1}$, the behavior of perturbation with time will not be exponential any more. However, a time derivative of \vec{z} at $t = 0$ will still be determined by the corresponding value of ω . In particular, the growth rate of the perturbation amplitudes will be given by $Re(\omega_s)$.

An information about $Re(\omega_s)$ can be used, first of all, to probe ill-posedness of a gauge. If $Re(\omega) \rightarrow \infty$ when $q \rightarrow \infty$ for at least one ω_n and at least in one direction e_k , it would be possible to construct a harmonic perturbation such that at $t = 0$ both the perturbation and any finite number of its derivatives will be less than any predetermined small number $\epsilon \ll 1$ but will become greater than any large predetermined number $A \gg 1$ after a finite period of time. This will mean, of course, that the gauge is ill-posed. If, on the other hand, the gauge is well-posed¹, then $Re(\omega_s)$ will determine the rate of growth of instabilities with time.

3 Gauge stability

In what follows, it will be convenient to distinguish between the three types of gauges. A gauge is called “fixed” if none of F_a in (2.12) involve γ_{ij} , and $F_a(x^b, \alpha, \beta_i) = 0$. We assume that (2.12) can be inverted and α, β_i can be expressed explicitly as functions of four-coordinates,

$$\alpha = \alpha(x^a) , \quad \beta_i = \beta_i(x^a) . \quad (3.1)$$

A fixed gauge does not change when coordinates are perturbed, $\frac{\partial \alpha}{\partial \psi_a} = \frac{\partial \beta_i}{\partial \psi_a} = 0$, and, as a result, $U = V_i = 0$. Equations (2.11) with $U = V_i = 0$ describe the evolution of coordinate perturbations for fixed gauges.

¹The absence of short-wave harmonic solutions with arbitrary large real increments is a necessary condition for well-posedness. Sufficient condition would require proving a continuous dependence of solutions on initial conditions for the entire 3+1 system.

A gauge is called “algebraic” or “local” if it can be expressed as a function of local values of γ_{ij} and its derivatives, $F_a \left(x^b, \alpha, \beta_i, \gamma_{ij}, \frac{\partial \gamma_{ij}}{\partial x^b}, \dots \right) = 0$, or

$$\alpha = \alpha(x^a, \gamma_{ij}, \frac{\partial \gamma_{ij}}{\partial x^b}, \dots) , \quad \beta_i = \beta_i(x^a, \gamma_{ij}, \frac{\partial \gamma_{ij}}{\partial x^b}, \dots) . \quad (3.2)$$

Fixed gauges are, of course, a particular case of algebraic gauges. Expressions for U and V_i for algebraic gauges reduce to

$$\begin{aligned} U &= 2 \frac{\partial \alpha}{\partial \gamma_{ij}} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + 2 \frac{\partial \alpha}{\partial \left(\frac{\partial \gamma_{ij}}{\partial x^a} \right)} \frac{\partial}{\partial x^a} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + \dots \\ V_k &= 2 \frac{\partial \beta_k}{\partial \gamma_{ij}} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + 2 \frac{\partial \beta_k}{\partial \left(\frac{\partial \gamma_{ij}}{\partial x^a} \right)} \frac{\partial}{\partial x^a} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_j}{\partial x_i} \right) + \dots \end{aligned} \quad (3.3)$$

Finally, a gauge is called “differential” or “non-local” if it cannot be reduced from (2.12) to an algebraic form. An algebraic gauge can always be expressed in a differential form by simply differentiating (3.2). Values of α and β_i for a differential gauge can only be expressed as a non-local functional of γ_{ij} .

3.1 Fixed gauges

To illustrate the approach outlined above, we begin with gauges (3.1) which are functions of spatial coordinates and time only. A synchronous gauge $\alpha = \alpha(t)$, $\beta_i = 0$ is an important member of this family. We will show below that it is the only well-posed fixed gauge. All other gauges (3.1) are ill posed.

For a fixed gauge, (2.11) become a first-order linear system of partial differential equations

$$\begin{aligned} \frac{\partial \psi_0}{\partial t} &= -D^{ij} \frac{\partial \psi_i}{\partial x^j} + C^a \psi_a , \\ \frac{\partial \psi_i}{\partial t} &= -\frac{\partial \psi_0}{\partial x^i} + E_i^a \psi_a . \end{aligned} \quad (3.4)$$

where

$$C^a = \Gamma_{00}^a + D^{ij} \Gamma_{ij}^a , \quad (3.5)$$

$$D^{ij} = \beta^i \beta^j , \quad (3.6)$$

and

$$E_k^a = 2\Gamma_{0k}^a . \quad (3.7)$$

For synchronous gauges (3.4) becomes

$$\begin{aligned} \frac{\partial \psi_0}{\partial t} &= \frac{1}{\alpha} \frac{\partial \alpha}{\partial t} \psi_0 \\ \frac{\partial \psi_i}{\partial t} &= -\frac{\partial \psi_0}{\partial x^i} + \gamma^{km} \frac{\partial \gamma_{mi}}{\partial t} \psi_k . \end{aligned} \quad (3.8)$$

A general solution of (3.8) is

$$\begin{aligned}\psi_0 &= \alpha(t)f_0 , \\ \psi_i &= \gamma_{ij} \left(f^i - \frac{\partial f_0}{\partial x^k} \int_0^t \alpha \gamma^{jk} dt \right) ,\end{aligned}\tag{3.9}$$

where $f_0(x^k), f^i(x^k)$ are smooth arbitrary function of spatial coordinates (see Problem 3 in Paragraph 97 of [31]). According to (3.9), there is a continuous dependence of solutions on initial conditions in the class of functions with continuous first derivatives. Thus a synchronous gauge is well posed.

Next, consider solutions of (3.4) in a short-wavelength limit. The matrix \hat{M} in (2.18) becomes

$$\hat{M} = \hat{M}_0 + \text{I}q\hat{M}_1\tag{3.10}$$

with

$$M_0 = \begin{pmatrix} C^0 & C^1 & C^2 & C^3 \\ E_1^0 & E_1^1 & E_1^2 & E_1^3 \\ E_2^0 & E_2^1 & E_2^2 & E_2^3 \\ E_3^0 & E_3^1 & E_3^2 & E_3^3 \end{pmatrix}, \quad \hat{M}_1 = \begin{pmatrix} 0 & D^{1k}e_k & D^{2k}e_k & D^{3k}e_k \\ e_1 & 0 & 0 & 0 \\ e_2 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 \end{pmatrix},\tag{3.11}$$

and (2.20) becomes a quartic equation for ω ,

$$\omega^4 + d_3\omega^3 + d_2\omega^2 + d_1\omega + d_0 = 0 ,\tag{3.12}$$

with coefficients being polynomial functions of q ,

$$\begin{aligned}d_0 &= d_{0,0} + d_{0,1} q + d_{0,2} q^2 , \\ d_1 &= d_{1,0} + d_{1,1} q + d_{1,2} q^2 , \\ d_2 &= d_{2,0} + d_{2,1} q + d_{2,2} q^2 , \\ d_3 &= d_{3,0} ,\end{aligned}\tag{3.13}$$

where $d_{i,j}$ are functions of γ_{ij} , α , β_i , and e_i . In what follows, we need expressions for

$$\begin{aligned}d_{0,2} &= (E_m^i E_i^j - E_n^i E_m^j) D^{km} e_k e_j + \frac{1}{2} (E_i^i E_j^j - E_j^i E_i^j) d_{2,2} , \\ d_{1,2} &= E_k^i D^{kj} e_i e_j - E_k^k d_{2,2} , \\ d_{2,1} &= -I(C^i + E_k^0 D^{ki}) e_i , \\ d_{2,2} &= D^{ij} e_i e_j , \text{ and} \\ d_{3,0} &= -C^0 - E_k^k .\end{aligned}\tag{3.14}$$

From (3.12) and (3.14) it is clear that the asymptotic behavior of roots of (3.12) with $q \rightarrow \infty$ must be $\omega = \sum \omega_k q^{k/m}$, with n, m integer. Values of m, n and coefficients w_k can be determined by substituting a power series expressions for ω into (3.12) and requiring that

terms with same powers of q cancel out. The result depends on whether some of $d_{i,j}$ are zero or not.

If $d_{2,2} \neq 0$, the asymptotic behavior of the roots is

$$\omega = \omega_1 q + \omega_0 + O\left(\frac{1}{q}\right), \quad (3.15)$$

with

$$(\omega_1)_{1,2,3,4} = \{0, 0, \pm i\sqrt{d_{2,2}}\}, \quad (3.16)$$

$$(\omega_0)_{1,2,3,4} = \left\{ -\frac{d_{1,2}}{2d_{2,2}} \pm \sqrt{\left(\frac{d_{1,2}}{2d_{2,2}}\right)^2 - \frac{d_{0,2}}{d_{2,2}}}, \quad -d_{3,0} \pm i\frac{d_{2,1}}{\sqrt{d_{2,2}}} + \frac{d_{1,2}}{d_{2,2}} \right\}. \quad (3.17)$$

If $d_{2,2} = 0$ but $d_{1,2} \neq 0$, the asymptotic behavior changes to

$$\omega = \omega_2 q^{2/3} + \omega_1 q^{1/3} + \omega_0 + O\left(\frac{1}{q^{1/3}}\right), \quad (3.18)$$

where

$$(\omega_2)_{1,2,3,4} = \text{Sign}(d_{1,2}) \cdot |d_{1,2}|^{1/3} \cdot \left\{ 0, 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\} \quad (3.19)$$

If both $d_{2,2} = d_{1,2} = 0$, the asymptotic behavior changes again, this time to

$$\omega = \omega_1 q^{1/2} + \omega_0 + O\left(\frac{1}{q^{1/2}}\right), \quad (3.20)$$

with

$$(\omega_1)_{1,2,3,4} = \pm \left(\frac{-d_{2,1} \pm \sqrt{-4d_{0,2} + (d_{2,1})^2}}{2} \right)^{1/2} \quad (3.21)$$

The asymptotic behavior is $\omega \sim O(1)$ if all three coefficients $d_{2,2} = d_{1,2} = d_{2,1} = 0$. In particular, for a synchronous gauge $\alpha = 1$, $\beta_i = 0$, the increments are determined by $\det(\gamma^{km} \frac{\partial \gamma_{mi}}{\partial t} - \omega \delta_i^m) = 0$.

For fixed algebraic gauges with zero shift we have $D^{ij} = 0$, $d_{2,2} = d_{1,2} = 0$, and from (3.21) the asymptotic behavior is

$$\omega_{1,2} = \pm q^{1/2} \left(1 - i \frac{C^i e_i}{|C^i e_i|} \right) \sqrt{\frac{|C^i e_i|}{2}} + O(1), \quad \omega_{3,4} = O(1). \quad (3.22)$$

C_i in (3.5) reduces for $\beta_i = 0$ to

$$C^i = \Gamma_{00}^i = -\alpha \gamma^{ik} \frac{\partial \alpha}{\partial x^k}. \quad (3.23)$$

Since e_i is a unit but otherwise arbitrary vector, there always will be harmonic solutions with $Re(\omega) \sim q^{1/2}$ and the gauge will be ill-posed unless lapse is spatially constant, $\alpha = \alpha(t)$.

Consider now gauges with $\beta_i \neq 0$. According to (3.16), $Re(w) \sim O(1)$ for all wavevectors not orthogonal to shift, $\beta^i e_i \neq 0$, so that these perturbations will not cause ill-posedness. However, it can be seen from (3.17) that for small $\beta_i \ll 1$ the increment $Re(\omega) \simeq \pm \beta^{-1} C^i e_i$ can become arbitrary large. Thus, a mode of perturbation unstable for $\beta_i = 0$ cannot be eliminated by applying a small shift in the direction of its propagation.

For wavevectors which are orthogonal to shift, $\beta^i e_i = 0$, we find $d_{1,2} = (E_k^i D^{kj} - E_k^j D^{ki}) e_i e_j = 0$, and the asymptotic behavior is again given by (3.22) but now with C^i determined by the general formula (3.5). If $C^i \beta_i \neq 0$, all orthogonal harmonic solutions will have $Re(\omega) \sim q^{1/2}$, and the gauge will be ill-posed. We now show that this is always the case, that is, C^i and β_i cannot be co-linear. Expressing Γ_{bc}^a in terms of three-dimensional quantities, we find that C^i can be written as

$$C^i = \gamma^{ik} \left(\frac{\partial \beta_k}{\partial t} + \frac{\partial(\beta^m \beta_m - \alpha^2)}{\partial x^k} \right) + f(x^a) \beta^i . \quad (3.24)$$

Thus, co-linearity of C^i and β_i requires that β_i must satisfy a system of partial differential equations

$$\frac{\partial \beta_k}{\partial t} - \frac{\partial(\alpha^2 - \gamma^{mn} \beta_n \beta_m)}{\partial x^k} = 0 . \quad (3.25)$$

These equations depend on γ^{nm} and, thus, β_i must depend on γ^{nm} as well. This contradicts, however, to our initial assumption (3.1) that α and β_i are explicit functions of spatial coordinates and do not change when γ^{nm} are perturbed. It is easy to see that the only solution of (3.25) independent of γ^{mn} is $\beta_i = 0$, and that this solution is possible if $\alpha = \alpha(t)$. We finally conclude that, unless shift is zero and lapse is spatially-constant, there always will be initially arbitrary small harmonic solutions of (3.11) which can be made arbitrary large after a finite period of time by an appropriate choice of q_i . That is, among fixed algebraic gauges (3.1) only synchronous gauges $\beta_i = 0$, $\alpha = \alpha(t)$ are well-posed. All other gauges (3.1) are ill-posed².

3.2 Algebraic gauges

As a next example, consider gauges with metric-dependent lapse and fixed shift,

$$\alpha = \alpha(x^a, \gamma_{ij}) , \quad \beta_i = \beta_i(x^a) \quad (3.26)$$

From (3.3) we obtain

$$U = 2 \frac{\partial \alpha}{\partial \gamma_{ij}} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_i}{\partial x^j} \right) , \quad V_i = 0 , \quad (3.27)$$

²In a one-dimensional case where both a metric and its perturbation are dependent on one spatial coordinate $x = x^1$, we have $\vec{e} = \{1, 0, 0\}$, and a non-zero shift along the x -coordinate leads to $\beta_i e_i \neq 0$ and to a well posed gauge. This however, is not a contradiction since coordinate perturbations orthogonal to gauge were not allowed.

and the coordinate perturbation equations (2.11) become

$$\begin{aligned}\frac{\partial\psi_0}{\partial t} &= -\left(\beta^i\beta^j + \frac{\partial\alpha^2}{\partial\gamma_{ij}}\right)\frac{\partial\psi_j}{\partial x_i} + \left(\Gamma_{00}^a + \left(\beta^i\beta^j + \frac{\partial\alpha^2}{\partial\gamma_{ij}}\right)\Gamma_{ij}^a\right)\psi_a \\ \frac{\partial\psi_i}{\partial t} &= -\frac{\partial\psi_0}{\partial x^i} + 2\Gamma_{0i}^a\psi_a.\end{aligned}\tag{3.28}$$

Comparing gauge perturbation equations (3.28) and (3.4) we observe that (3.28) becomes identical to (3.4) if the definition of D^{ij} is changed from (3.6) to

$$D^{ij} = \beta^i\beta^j + \frac{\partial\alpha^2}{\partial\gamma_{ij}}.\tag{3.29}$$

In what follows, we assume that $\frac{\partial\alpha^2}{\partial\gamma_{ij}}$ and, thus, D^{ij} are symmetric. We obtain a dispersion relation similar to (3.12), and, after taking a limit of $q \rightarrow \infty$, get an asymptotic behavior of wavenumbers ω which dependence on coefficients $d_{i,j}$ is similar to that described by equations (3.15-3.21). Now, however, $D^{ij} \neq \beta^i\beta^j$ and thus $d_{2,2} = D^{ij}e_ie_j$ may be both positive, zero, and negative. For $d_{2,2} < 0$ we have two roots with $Re(\omega) \sim q$, and the gauge is ill-posed. If $d_{2,2} = 0$, $d_{1,2} \neq 0$, we see from (3.19) that one root will have $Re(\omega) \sim q^{2/3}$ and the gauge is again ill-posed. If $d_{2,2} = 0$, $d_{1,2} = 0$, $d_{2,1} \neq 0$, then one root has $Re(\omega) \sim q^{1/2}$ and the gauge is ill-posed as well. Having $d_{2,2} = d_{1,2} = d_{2,1} = 0$ and, thus, the asymptotic behavior $\omega \sim O(1)$ for all four roots is impossible. A reasoning similar to that used for fixed algebraic gauges above shows that this requires β_i to be functions of the metric and contradicts to the assumption (3.26). We finally conclude that gauges (3.26) are ill-posed unless $D^{ij}e_ie_j$ is strictly positive, i.e., unless

$$D^{ij}e_ie_j = \left(\beta^i\beta^j + \frac{\partial\alpha^2}{\partial\gamma_{ij}}\right)e_ie_j > 0\tag{3.30}$$

is satisfied for every e_i .

One simple gauge of type (3.26) which has been used in a variety of works is a gauge which depends on the determinant of a three-metric,

$$\alpha = \alpha(x^a, \gamma), \quad \gamma = \det(\gamma_{ij}).\tag{3.31}$$

For this gauge

$$D^{ij} = \beta^i\beta^j + \frac{\partial\alpha^2}{\partial\gamma}\gamma^{ij}\tag{3.32}$$

$(d\gamma = \gamma\gamma^{ij}d\gamma_{ij})$, and, since γ^{ij} is positive definite, the gauge will be ill-posed unless

$$\frac{\partial\alpha^2}{\partial\gamma} > 0.\tag{3.33}$$

More general metric-dependent gauges may be investigated in a similar way. For example, gauges with both α and β_i functions of the metric,

$$\alpha = \alpha(x^a, \gamma_{ij}), \quad \beta_i = \beta_i(x^a, \gamma_{ij}),\tag{3.34}$$

have both $U \neq 0$ and $V_i \neq 0$,

$$\begin{aligned} U &= 2 \frac{\partial \alpha}{\partial \gamma_{ij}} \left(\Gamma_{ij}^c \psi_c - \frac{\partial \psi_i}{\partial x^j} \right) , \\ V_i &= 2 \frac{\partial \beta_i}{\partial \gamma_{jk}} \left(\Gamma_{jk}^c \psi_c - \frac{\partial \psi_j}{\partial x^k} \right) , \end{aligned} \quad (3.35)$$

and the resulting coordinate perturbation equations become more complicated

$$\begin{aligned} \frac{\partial \psi_0}{\partial t} &= -D^{ij} \frac{\partial \psi_i}{\partial x^j} + C^a \psi_a , \\ \frac{\partial \psi_i}{\partial t} &= -\frac{\partial \psi_0}{\partial x^i} - G_i^{jk} \frac{\partial \psi_j}{\partial x^k} + E_i^a \psi_a . \end{aligned} \quad (3.36)$$

Here

$$\begin{aligned} C^a &= \Gamma_{00}^a + D^{ij} \Gamma_{ij}^a , \\ D^{ij} &= \beta^i \beta^j + \frac{\partial \alpha^2}{\partial \gamma_{ij}} - 2 \beta^n \frac{\partial \beta_n}{\partial \gamma_{ij}} , \\ E_k^a &= 2 \Gamma_{0k}^a - 2 \Gamma_{ij}^a \frac{\partial \beta_k}{\partial \gamma_{ij}} , \\ G_k^{ij} &= -2 \frac{\partial \beta_k}{\partial \gamma_{ij}} . \end{aligned} \quad (3.37)$$

The matrix \hat{M}_0 remains the same, but \hat{M}_1 now becomes

$$\hat{M}_1 = \begin{pmatrix} 0 & D^{1k} e_k & D^{2k} e_k & D^{3k} e_k \\ e_1 & G_1^{1k} e_k & G_1^{2k} e_k & G_1^{3k} e_k \\ e_2 & G_2^{1k} e_k & G_2^{2k} e_k & G_2^{3k} e_k \\ e_3 & G_3^{1k} e_k & G_3^{2k} e_k & G_3^{3k} e_k \end{pmatrix} . \quad (3.38)$$

The dispersion relation is still a quartic equation (3.12) but its coefficients now are forth-order polynomials in q . The analysis of the roots $\omega_{1,2,3,4}$ is a more complicated algebraic problem. In general, (3.34) may be either ill- or well-posed depending on the functional form of α and β_i .

It must be stressed here that a well-posedness is not a guarantee of gauge stability. Metric-dependent gauges satisfying (3.30) are generally unstable with the increment of instability $Re(\omega) = \omega_0$ given by formulas (3.17) with coefficients determined according to (3.5), (3.7), (3.14), and (3.29). In particular, it can be shown using these formulas that the gauge (3.31) can be unstable for a wide range of background solutions. When (3.34) is well-posed, it can be both stable and unstable depending on a particular background solution.

A class of gauges often considered in the literature is [5]

$$\frac{\partial \alpha}{\partial t} - \beta^i \frac{\partial \alpha}{\partial x^i} = -\alpha^2 f(\alpha) \text{tr}(K_{ij}) \quad (3.39)$$

where $f(\alpha)$ is an arbitrary function. It was found that $f \geq 0$ is a necessary condition for hyperbolicity of first-order 3+1 formulations of GR introduced in that paper. The gauge (3.39) is, in fact, equivalent to an algebraic gauge [8]

$$\sqrt{\gamma} = F(\alpha) \quad (3.40)$$

with $f = \alpha F \left(\frac{\partial F}{\partial \alpha} \right)^{-1}$, and contains as its members such gauges as a harmonic slicing ($f = 1$) and a "1+log" slicing ($f = 1/\alpha$). One can see that the condition $f > 0$ derived from the analysis of hyperbolicity of the entire 3+1 system of equations is equivalent to our condition of well-posedness $\frac{\partial F}{\partial \alpha} > 0$ derived from the analysis of gauge modes alone.

A more general family of first-order 3+1 systems has been derived in [19] using a gauge

$$\log(\alpha g^{-\sigma}) = Q(x^a) , \quad \beta_i = \beta_i(x^a) . \quad (3.41)$$

This gauge belongs to a family of algebraic gauges (3.26) as well. It was found in [19] that having a metric-dependent, densitized lapse with $\sigma > 0$ is a necessary condition for a hyperbolicity of 3+1 systems considered in that work. Again, it is easy to see that the requirement $\sigma > 0$ is equivalent to the condition of well-posedness derived in this paper from the analysis of gauge instabilities.

3.3 Differential gauges

As an example of a differential gauge, we consider a parabolic extension of a well known maximal slicing gauge $\gamma^{ij} \nabla_i \nabla_j \alpha = K^{ij} K_{ij} \alpha$, $\beta_i = 0$ [36, 37],

$$\frac{\partial \alpha}{\partial t} = \frac{1}{\epsilon} (\gamma^{ij} \nabla_i \nabla_j \alpha - K^{ij} K_{ij} \alpha) , \quad \beta_i = 0 , \quad (3.42)$$

where $\epsilon > 0$ is a constant. We follow the same general procedure as that used above for fixed and algebraic gauges. Expanding covariant derivatives in (3.42) and taking into account that, for $\beta_i = 0$, the extrinsic curvature $K_{ij} = -\frac{1}{2\alpha} \pi_{ij}$, we rewrite (3.42) as

$$F_0 \equiv -\epsilon \frac{\partial \alpha}{\partial t} + \gamma^{ij} \frac{\partial^2 \alpha}{\partial x^i \partial x^j} - \gamma^{ij} \lambda_{ij}^k \frac{\partial \alpha}{\partial x^k} - \frac{\pi^{ij} \pi_{ij}}{4\alpha} = 0 , \quad \beta_i = 0 , \quad (3.43)$$

and find the derivatives of F_0 ,

$$\begin{aligned} \frac{\partial F_0}{\partial \alpha} &= \frac{\pi_{kl} \pi^{kl}}{4\alpha^2} , \quad \frac{\partial F_0}{\partial (\frac{\partial \alpha}{\partial x^i})} = -\gamma^{kl} \lambda_{kl}^i , \quad \frac{\partial F_0}{\partial (\frac{\partial^2 \alpha}{\partial x^i \partial x^j})} = \gamma^{ij} , \quad \frac{\partial F_0}{\partial (\frac{\partial \alpha}{\partial t})} = -\epsilon , \\ \frac{\partial F_0}{\partial \gamma_{ij}} &\equiv A^{ij} = \frac{\gamma_{kl} \pi^{ik} \pi^{jl}}{2\alpha} - \gamma^{ik} \gamma^{jl} \frac{\partial^2 \alpha}{\partial x^k \partial x^l} + (\gamma^{ik} \gamma^{jl} \lambda_{kl}^n + \gamma^{kl} \gamma^{in} \lambda_{kl}^j) \frac{\partial \alpha}{\partial x^n} , \\ \frac{\partial F_0}{\partial (\frac{\partial \gamma_{ij}}{\partial x^k})} &\equiv B^{ij,k} = -\frac{1}{2} \frac{\partial \alpha}{\partial x^n} (\gamma^{ik} \gamma^{jn} + \gamma^{jk} \gamma^{in} - \gamma^{ij} \gamma^{kn}) , \quad \text{and} \quad \frac{\partial F_0}{\partial \pi_{ij}} = -\frac{\pi^{ij}}{2\alpha} . \end{aligned} \quad (3.44)$$

Substituting (3.44) into (2.13), and combining (2.13) with (2.11) we obtain the system of five quasi-linear partial differential equations

$$\begin{aligned}
\frac{\partial \psi_0}{\partial t} &= \alpha U + \Gamma_{00}^c \psi_c , \\
\frac{\partial \psi_i}{\partial t} &= -\frac{\partial \psi_0}{\partial x^i} + 2\Gamma_{0i}^c \psi_c , \\
\epsilon \frac{\partial U}{\partial t} &= \gamma^{ij} \nabla_i \nabla_j U + \frac{\pi_{ij} \pi^{ij}}{4\alpha^2} U - \frac{\pi^{ij}}{\alpha} \frac{\partial^2 \psi_0}{\partial x^i \partial x^j} - 2B^{ij,k} \frac{\partial^2 \psi_i}{\partial x^j \partial x^k} + C^i \frac{\partial \psi_0}{\partial x_i} + D^{ij} \frac{\partial \psi_i}{\partial x_j} + J^c \psi_c ,
\end{aligned} \tag{3.45}$$

where

$$\begin{aligned}
C^k &= \frac{\pi^{ij}}{\alpha} \Gamma_{ij}^k + 2\frac{\pi^{ik}}{\alpha} \Gamma_{0i}^0 + 2B^{ij,k} \Gamma_{ij}^0 , \quad D^{ij} = \frac{2\Gamma_{0k}^i \pi^{kj}}{\alpha} - 2A^{ij} + 2B^{lk,j} \Gamma_{lk}^i , \text{ and} \\
J^c &= 2A^{ij} \Gamma_{ij}^c - \frac{\pi^{ij}}{\alpha} \left(\frac{\partial \Gamma_{ij}^c}{\partial t} + \Gamma_{ij}^0 \Gamma_{00}^c + 2\Gamma_{ij}^k \Gamma_{0k}^c - 2\frac{\partial \Gamma_{0i}^c}{\partial x^j} \right) + 2B^{ij,k} \frac{\partial \Gamma_{ij}^c}{\partial x^k} .
\end{aligned} \tag{3.46}$$

The dispersion relation becomes

$$\det \begin{pmatrix} \Gamma_{00}^0 - \omega & \Gamma_{00}^1 & \Gamma_{00}^2 & \Gamma_{00}^3 & \alpha \\ 2\Gamma_{01}^0 + \text{I}e_1 q & 2\Gamma_{01}^1 - \omega & 2\Gamma_{01}^2 & 2\Gamma_{01}^3 & 0 \\ 2\Gamma_{02}^0 + \text{I}e_2 q & 2\Gamma_{02}^1 & 2\Gamma_{02}^2 - \omega & 2\Gamma_{02}^3 & 0 \\ 2\Gamma_{03}^0 + \text{I}e_3 q & 2\Gamma_{03}^1 & 2\Gamma_{03}^2 & 2\Gamma_{03}^3 - \omega & 0 \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} - \omega \end{pmatrix} = 0 , \tag{3.47}$$

where

$$\begin{aligned}
M_{40} &= \frac{1}{\epsilon} \left(J^0 - \text{I}C^i e_i q + \frac{\pi^{ij} e_i e_j}{\alpha} q^2 \right) , \\
M_{4i} &= \frac{1}{\epsilon} \left(J^i - \text{I}D^{ij} e_j q + 2B^{ij,l} e_j e_l q^2 \right) , \text{ and} \\
M_{44} &= \frac{1}{\epsilon} \left(\frac{\pi_{ij} \pi^{ij}}{4\alpha^2} + \text{I}\gamma^{ij} \lambda_{ij}^k e_k q - \gamma^{ij} e_i e_j q^2 \right) .
\end{aligned} \tag{3.48}$$

The dispersion relation (3.47) is a fifth-order equation with respect to ω ,

$$-\omega^5 + d_4 \omega^4 + d_3 \omega^3 + d_2 \omega^2 + d_1 \omega + d_0 = 0 , \tag{3.49}$$

where

$$\begin{aligned}
d_4 &= d_{4,0} + d_{4,1}q + d_{4,2}q^2 , \\
d_3 &= d_{3,0} + d_{3,1}q + d_{3,2}q^2 , \\
d_2 &= d_{2,0} + d_{2,1}q + d_{2,2}q^2 + d_{2,3}q^3 , \\
d_1 &= d_{1,0} + d_{1,1}q + d_{1,2}q^2 + d_{1,3}q^3 , \\
d_0 &= d_{0,0} + d_{0,1}q + d_{0,2}q^2 + d_{0,3}q^3 ,
\end{aligned} \tag{3.50}$$

and coefficients $d_{i,j}$ are functions of an unperturbed metric g_{ab} . In what follows, we need explicit expressions for

$$\begin{aligned}
d_{4,2} &= \epsilon^{-1} \gamma^{ij} e_i e_j , \\
d_{4,1} &= \text{I} \epsilon^{-1} \gamma^{ij} \lambda_{ij}^k e_k , \\
d_{2,3} &= \text{I} \left(2\epsilon^{-1} \alpha B^{ij,k} e_i e_j e_k + d_{4,2} \Gamma_{00}^k e_k \right) = -\text{I} d_{4,2} \Gamma_{00}^k e_k = -\text{I} \alpha \frac{\partial \alpha}{\partial x^n} d_{4,2} \gamma^{nk} e_k .
\end{aligned} \tag{3.51}$$

From (3.49), (3.50) we obtain the following asymptotic behavior of the roots.

$$\begin{aligned}
\omega_1 &= -d_{4,2} q^2 - d_{4,1} q + O(1) , \\
\omega_{2,3} &= \pm q^{1/2} \sqrt{-\frac{d_{2,3}}{d_{4,2}}} + O(1) = \pm \left(1 - \text{I} \frac{\Gamma_{00}^k e_k}{|\Gamma_{00}^k e_k|} \right) q^{1/2} \sqrt{\frac{|\Gamma_{00}^k e_k|}{2}} + O(1) , \\
\omega_{4,5} &= O(1) .
\end{aligned} \tag{3.52}$$

Since γ^{ij} is positive definite, we have $d_{4,2} > 0$, and thus $\text{Re}(\omega_1) < 0$ in the limit of $q \rightarrow \infty$. This root does not cause an instability. However, one of the roots $\omega_{2,3}$ has a positive $\text{Re}(\omega)$ growing with q as

$$\text{Re}(\omega) \propto q^{1/2} \sqrt{\frac{\Gamma_{00}^k e_k}{2}} = q^{1/2} \sqrt{\frac{1}{2} \alpha \gamma^{ik} e_k \left| \frac{\partial \alpha}{\partial x^k} \right|} .$$

Since the direction of e_i is arbitrary, a maximal slicing gauge will be ill posed unless α is spatially constant. The result we obtain for a maximal slicing and its parabolic extension is exactly the same as that we found earlier for fixed algebraic gauges (see (3.22)). It is easy to see that in both cases the asymptotic $q^{1/2}$ -behavior of the roots comes from the presence of $\Gamma_{00}^i \psi_i$ term in the right-hand side of the first equation in either (3.4) or (3.45). Differentiating first equation with respect to time and substituting second equation into it, we obtain in both cases a parabolic-like equation for ψ_0

$$\frac{\partial^2 \psi_0}{\partial t^2} = \Gamma_{00}^i \frac{\partial \psi_0}{\partial x^i} + \text{other terms}$$

which is ill-posed if initial conditions are specified at $t = \text{const.}$ (see also next section).

4 Physical meaning of gauge instabilities

4.1 Gauge transformations in a flat spacetime

To illustrate our analysis of gauge stability, we must consider now the entire evolution part of the Einstein equations and compare its stability with that of the system (2.11), (2.13). We will use an ADM 3+1 system with metric γ_{ij} and extrinsic curvature K_{ij} as unknown variables,

$$\begin{aligned}\frac{\partial \gamma_{ij}}{\partial t} &= -2\alpha K_{ij} + \nabla_j \beta_i + \nabla_i \beta_j , \\ \frac{\partial K_{ij}}{\partial t} &= -\nabla_i \nabla_j \alpha + \alpha (R_{ij} + K K_{ij} - 2K_{im} K_j^m) \\ &\quad + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{jm} \nabla_i \beta^m ,\end{aligned}\tag{4.1}$$

and will consider a simple class of one-dimensional solutions of (4.1)

$$\gamma_{ij} = \text{diag}(\gamma, 1, 1) , K_{ij} = \text{diag}(K, 0, 0)\tag{4.2}$$

with γ and K dependent on t and $x \equiv x^1$ only. In vector notations, (4.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}(\mathbf{u}) + \hat{B}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x}\tag{4.3}$$

where

$$\mathbf{u} = \begin{pmatrix} \gamma \\ K \end{pmatrix} , \quad \mathbf{A} = \begin{pmatrix} -2\alpha K + 2\frac{\partial \beta}{\partial x} \\ -\frac{\partial^2 \alpha}{\partial x^2} - \frac{\alpha K^2}{\gamma} + \frac{2K}{\gamma} \frac{\partial \beta}{\partial x} \end{pmatrix} , \quad \hat{B} = \begin{pmatrix} -\frac{\beta}{\gamma} & 0 \\ \frac{1}{2\gamma} \frac{\partial \alpha}{\partial x} - \frac{2K\beta}{\gamma^2} & \frac{\beta}{\gamma} \end{pmatrix} .\tag{4.4}$$

One can easily verify that all components of the Riemann tensor are identically zero for any metric (4.2) satisfying (4.3). That is, (4.2) describes a flat spacetime in non-Galilean coordinates³. One can also verify that the constraint equations are automatically satisfied for any metric (4.2). Therefore, (4.3) can be unstable only with respect to gauge instabilities.

Eigenvectors and eigenvalues of \hat{B} in (4.3) are

$$\lambda = \pm \frac{\beta}{\gamma} , \quad \mathbf{u} = \begin{pmatrix} 4\beta\gamma/(4\beta K - \gamma \frac{\partial \alpha}{\partial x}) \\ 1 \end{pmatrix} , \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} .\tag{4.5}$$

³The only nontrivial component of the Riemann tensor in this case is R_{1010} . Since choice of α, β is a choice of a coordinate system, it is sufficient to show that $R_{1010} \equiv 0$ in a coordinate system with $\alpha = 1, \beta = 0$. In this case, non-zero Cristoffels are

$$\Gamma_{01}^1 = \Gamma_{01}^1 = \frac{1}{2\gamma} \frac{\partial \gamma}{\partial t} , \quad \Gamma_{11}^0 = \frac{1}{2} \frac{\partial \gamma}{\partial t} , \quad \Gamma_{11}^1 = \frac{1}{2\gamma} \frac{\partial \gamma}{\partial x} ,$$

and

$$R_{0101} = -\frac{1}{2} \frac{\partial^2 g_{11}}{\partial t^2} + g_{11} \Gamma_{10}^1 \Gamma_{01}^1 = -\frac{1}{2} \frac{\partial^2 \gamma}{\partial t^2} + \frac{1}{4\gamma} \left(\frac{\partial \gamma}{\partial t} \right)^2 \equiv 0 .$$

For $\beta \neq 0$, there is a complete set of eigenvalues and eigenvectors, so that (4.3) is strongly hyperbolic and well posed. For $\beta = 0$, a complete set does not exist, and the system reduces to a parabolic equation

$$\frac{\partial^2 \gamma}{\partial t^2} = \left(\frac{\alpha}{\gamma} \frac{\partial \alpha}{\partial x} \right) \frac{\partial \gamma}{\partial x} + 2\alpha^2 \frac{K^2}{\gamma} - 2 \frac{\partial \alpha}{\partial t} K + 2\alpha \frac{\partial^2 \alpha}{\partial x^2} \quad (4.6)$$

which is ill-posed (a parabolic equation $\frac{\partial a}{\partial \tilde{x}} = \frac{\partial^2 a}{\partial \tilde{y}^2}$ is ill posed if initial conditions are defined at $\tilde{x} = \text{const}$, e.g., [28]).

Analysis of the principal part $\hat{B} \frac{\partial \mathbf{u}}{\partial x}$ is not sufficient, however, for investigation of instabilities of (4.3). We need to carry out a full stability analysis of this system. A linearized version of (4.3) is

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} = \hat{C}(\mathbf{u}) \bar{\mathbf{u}} + \hat{B}(\mathbf{u}) \frac{\partial \bar{\mathbf{u}}}{\partial x} \quad (4.7)$$

where \mathbf{u} is now an unperturbed base solution, $\bar{\mathbf{u}}$ is a vector of perturbations, and

$$\hat{C} = \frac{\partial \mathbf{A}}{\partial \mathbf{u}} + \frac{\partial \hat{B}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} = \begin{pmatrix} \frac{\alpha K^2}{\gamma^2} - \frac{\partial \alpha}{\partial x} \frac{1}{2\gamma^2} \frac{\partial \gamma}{\partial x} - \frac{\beta}{\gamma^2} \frac{\partial K}{\partial x} - \frac{2K}{\gamma^2} \frac{\partial \beta}{\partial x} + \frac{4K\beta}{\gamma^3} \frac{\partial \gamma}{\partial x} & -2\alpha \\ \frac{2}{\gamma} \frac{\partial \beta}{\partial x} - \frac{\beta}{\gamma^2} \frac{\partial \gamma}{\partial x} - \frac{2\alpha K}{\gamma} \end{pmatrix}. \quad (4.8)$$

For harmonic solutions

$$\bar{\mathbf{u}} \propto \exp(\omega t - Iq x), \quad (4.9)$$

in the limit $q \rightarrow \infty$ we then obtain a dispersion relation

$$||\hat{C} - Iq\hat{B} - \hat{I}\omega|| = \omega^2 + d_1\omega + d_0 = 0, \quad (4.10)$$

with

$$\begin{aligned} d_1 &= 2 \left(\frac{\alpha K}{\gamma} - \frac{1}{\gamma} \frac{\partial \beta}{\partial x} \right), \\ d_0 &= d_{0,0} + Id_{0,1}q + d_{0,2}q^2, \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} d_{0,0} &= \left(\frac{\beta}{\gamma^2} \frac{\partial \gamma}{\partial x} \right) \left(\frac{2}{\gamma} \frac{\partial \beta}{\partial x} - \frac{2\beta}{\gamma^2} \frac{\partial \gamma}{\partial x} - \frac{2\alpha K}{\gamma} \right) \\ &\quad + 2\alpha \left(\frac{\alpha K^2}{\gamma^2} - \frac{\partial \alpha}{\partial x} \frac{1}{2\gamma^2} \frac{\partial \gamma}{\partial x} - \frac{\beta}{\gamma^2} \frac{\partial K}{\partial x} - \frac{2K}{\gamma^2} \frac{\partial \beta}{\partial x} + \frac{4K\beta}{\gamma^3} \frac{\partial \gamma}{\partial x} \right), \\ d_{0,1} &= \frac{2\beta}{\gamma} \left(\frac{1}{\gamma} \frac{\partial \beta}{\partial x} - \frac{\beta}{\gamma^2} \frac{\partial \gamma}{\partial x} + \frac{\alpha K}{\gamma} \right) - \frac{\alpha}{\gamma} \frac{\partial \alpha}{\partial x}, \\ d_{0,2} &= \frac{\beta^2}{\gamma^2}. \end{aligned} \quad (4.12)$$

For $\beta \neq 0$, $d_{0,2} \neq 0$ and we get an asymptotic behavior

$$w = \pm I q \frac{\beta}{\gamma} \pm \frac{\alpha}{2\beta} \frac{\partial \alpha}{\partial x} \pm \frac{\beta}{\gamma^2} \frac{\partial \gamma}{\partial x} - \frac{\alpha K}{\gamma} + \frac{1}{\gamma} \frac{\partial \beta}{\partial x} \mp \left(\frac{\alpha K}{\gamma} + \frac{1}{\gamma} \frac{\partial \beta}{\partial x} \right) + O\left(\frac{1}{q}\right) \quad (4.13)$$

We see that although the system is well-posed, it has $Re(\omega) \sim O(1)$, and is unstable with respect to small perturbations (unless $\frac{\partial \beta}{\partial x} = \frac{\partial \gamma}{\partial x} = \frac{\partial \alpha}{\partial x} = 0$, and $K > 0$). If $\frac{\partial \alpha}{\partial x} \neq 0$, the increment of the instability $Re(\omega) \sim \beta^{-1} \rightarrow \infty$ when $\beta \rightarrow 0$.

For $\beta = 0$ and $\frac{\partial \alpha}{\partial x} \neq 0$, we have $d_{0,2} = 0$, $d_{0,1} \neq 0$, and the asymptotic behavior is

$$\omega = \pm(1 + I) q^{1/2} \sqrt{\frac{\alpha}{2\gamma} \frac{\partial \alpha}{\partial x}} + O(1) \quad (4.14)$$

which shows that $Re(\omega) \sim q^{1/2}$ and the system is ill-posed. The conclusions agree with the results of gauge stability analysis of the previous section.

4.2 Instability of a synchronous coordinate system

Synchronous coordinate systems are formed by acceleration-free test particles. It is well known that in such systems the metric determinant $|\gamma_{ij}|$ will vanish after a finite time because the time-lines of a reference frame will necessarily intersect one another on a certain caustic hypersurface [31]. This is correct in both flat and curved spacetimes. On a caustic, one of the principal values of the metric tensor vanishes, whereas the corresponding contravariant component tends to infinity. In the vicinity of a caustic, using an arbitrariness in the selection of spatial coordinates one can write a general four-metric as [32]

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{np} = \gamma_{np} = a_{np}, \quad g_{n3} = \gamma_{n1} = \tau^2 a_{n1}, \quad g_{11} = \gamma_{11} = \tau^2 a_{11} \quad (4.15)$$

where $\tau = t - x^1$, indices n, p take values 2, 3, and a_{ij} are non-singular functions of x^2, x^3 , and t . Coefficients a_{ik} are connected by a single relation which is a consequence of the Einstein equations (see Appendix in [32]). Equation (4.15) determines a general “quadratic” character of approaching a caustic in a synchronous gauge. In (4.15), some of the contravariant components of g^{ab} tend to infinity as τ^{-2} . Because of that, the process described by (4.15) is sometimes called a “blow up” instability.

It is important to distinguish between the formation of a caustic (blow up) and the instability of a synchronous gauge with respect to small perturbations considered in this paper. To clarify this important point, let us consider the following example. Consider one-dimensional solutions of (4.3) for $\alpha = 1$ and $\beta = 0$. The equations (4.3) become

$$\frac{\partial \gamma}{\partial t} = -2K, \quad \frac{\partial K}{\partial t} = -\frac{K^2}{\gamma}, \quad (4.16)$$

and a general solution of (4.16) is⁴

$$\gamma = (A(x) + B(x)t)^2, \quad K = -B(x)(A(x) + B(x)t). \quad (4.17)$$

⁴Physical meaning of A and B in this solution is discussed in the Appendix

If at some point we have $A > 0$, $B < 0$, a caustic will form and both γ and K will become zero at some future time $t_c = -A/B > 0$. Note that the process of caustic formation may be global if $\frac{\partial A}{\partial x}, \frac{\partial B}{\partial x} = 0$ or local otherwise. Using (4.17) it is easy to write the deviation of γ , $\delta\gamma$ as a function of perturbations of A and B ,

$$\delta\gamma = 2(A + Bt)(A\delta A + B\delta Bt) , \quad (4.18)$$

where $\delta A = \Delta A/A$, $\delta B = \Delta B/B$, and $\delta A, \delta B \ll 1$. This is a weak instability described by a power function of t instead of an exponential function⁵. Both the formation of a caustic and the instability are due to the presence of initial velocities of the reference frame with respect to an inertial Galilean frame. However, the caustic is due to the intersection of converging trajectories of test particles, whereas the instability refers to a divergence of test particles from unperturbed time-lines.

4.3 Gauge instabilities in accelerating systems

Let us now discuss a physical meaning of ill-posedness and instability of non-synchronous gauges. Consider first a Rindler reference frame [38] where this meaning is most clear. Let T, X^1, X^2, X^3 be a Cartesian coordinate system in a Minkowski space-time with an interval

$$ds^2 = -dT^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \quad (4.19)$$

The Rindler frame is constructed from the world lines of particles moving with accelerations along an X-axis of this coordinate system in such a way that the accelerations are constant in proper time of the frame. Such a frame is "rigid" in the sense that It does not deform in proper time. For an accelerating frame to be rigid, accelerations of the particles must be different with respect to the inertial frame. If accelerations of particles with different X^1 and same time T are equal, the distance between particles measured in the inertial coordinate frame will stay constant. Due to a Lorentz contraction, distances measured in the accelerating frame will depend on the velocity of the frame, and will change with time.

The Rindler frame can be described using coordinates x, t (we omit two other spatial coordinates for brevity) as

$$T = x \sinh(gt), \quad X^1 = x \cosh(gt) , \quad (4.20)$$

where g is a constant. In these coordinates the interval becomes

$$ds^2 = -(gx)^2 dt^2 + dx^2 , \quad (4.21)$$

and it is evident that the geometry of the co-moving space in this frame is time-independent. Coordinate lines of t , $T^2 = X^2 - x^2$, are world lines of uniformly accelerated particles. An acceleration a measured by a co-moving observer in an orthogonal coordinate system x^a is [39]

$$a = \sqrt{\gamma_{ik} a^i a^k} , \quad a^i = \Gamma_{00}^i / g_{00} . \quad (4.22)$$

⁵A formal stability analysis gives in this case a growth rate of perturbations $\omega = -K/\gamma$ comparable to that of an unperturbed solution. As it was discussed in Section 3, (2.19) is not valid in this case.

For a system with acceleration along the X^1 -axis we obtain

$$a = \frac{1}{\alpha\sqrt{\gamma_{11}}} \left| \frac{\partial\alpha}{\partial x} \right| \quad (4.23)$$

and for a Rindler frame

$$a_R = \frac{1}{x} \quad (4.24)$$

In a Rindler frame, the coordinate x is equal to a physical distance $l = \int_0^x \sqrt{\gamma_{11}} dx$ from the point $x = 0$ where physical acceleration is infinite. Therefore, (4.24) can be written in terms of quantities which have a direct physical meaning,

$$a_R = \frac{1}{l} . \quad (4.25)$$

The dependence (4.25) of a physical acceleration on a physical distance provides the rigidity of a Rindler frame.

The perturbation equations (4.7) for a Rindler frame can be reduced to a single equation

$$\frac{\partial^2 \bar{\gamma}}{\partial t^2} = -g^2 x \frac{\partial \bar{\gamma}}{\partial x} \quad (4.26)$$

where $\bar{\gamma}$ is a perturbation of γ_{11} . Consider the following exact solutions of (4.26) (modes of perturbation):

1. $\bar{\gamma} = \text{const} \ll 1$. This corresponds to a uniform change of the length scale along the x -axis. Such a perturbation changes both the acceleration $a = \frac{1}{x}(1 + \bar{\gamma})^{-1/2}$ and the physical distance $l = \int_0^x (1 + \bar{\gamma})^{1/2} dx = (1 + \bar{\gamma})^{1/2} x$ in such a way that still $a = 1/l$. The condition (4.24) is not violated and the frame remains rigid.
2. $\bar{\gamma} = At$, $At \ll 1$. The perturbation does not depend on x and describes a deformation of the frame with constant velocities with respect to a Rindler frame. Formally this is an instability but of the type similar to that existing in synchronous gauges. As in the previous case, it can be shown that the relation $a = 1/l$ holds.
3. $\bar{\gamma} = x^{-A^2/g^2} (c_1 \sinh(At) + c_2 \cosh(At))$. This time the initial perturbation depends on x and grows exponentially. Suppose $c_2 > 0$ so that $\bar{\gamma} > 0$, then $\frac{\partial \bar{\gamma}}{\partial x} < 0$. The acceleration is again $a(x) = \frac{1}{x}(1 + \bar{\gamma}(x))^{-1/2}$. The physical distance, however, is $l = \int_0^x (1 + \bar{\gamma})^{1/2} dx = x(1 + \bar{\gamma}(x'))^{1/2}$ with some $x' < x$. As a result, $a(x) > 1/l(x)$ is now greater than that required to maintain rigidity, and the deformation will grow with time. A perturbation $\bar{\gamma} = x^{A^2/g^2} (c_1 \sin(At) + c_2 \cos(At))$ is also a solution of (4.26), but now $\frac{\partial \bar{\gamma}}{\partial x} < 0$ and this particular perturbation is stable.

We see that the reason for the instability is a mismatch between the deformation of the frame and the prescribed acceleration of particles which leads to further frame deformation. We can imagine, for example, a deformation with $\bar{\gamma} = 0$ everywhere for x less than some

value x_0 , $\bar{\gamma} = \bar{\gamma}_0 = \text{const} > 0$ at $x > x_1 > x_0$ and $\bar{\gamma}$ increasing linearly between x_0 and x_1 . The difference between accelerations at x_0 and x_1 will be finite, but by tending $x_1 \rightarrow x_0$ we can make $\gamma(t)/\gamma(0)$ between the neighboring points x_0 and x_1 to increase with any desired rate. This corresponds to the mathematical property of ill-posedness of fixed gauges with spatially non-uniform lapse. It is also obvious that when α and γ depend on t and x , the instability will mean a deviation from an already deforming reference frame.

In one-dimensional case, any metric with non-zero shift can be obtained from a metric with $\beta = 0$ by a transformation of a time coordinate $\bar{t} = \bar{t}(t, x)$. Such a transformation does not change the accelerations of test particles which remain “chronometric invariants” [40]. However, due to a non-zero shift, perturbations are now “advected”, that is, their coordinates change continuously whereas accelerations are still prescribed at fixed coordinates. As a result, $\gamma(x)$ cannot grow with an arbitrary rate. An apparent growth of the instability at fixed x will decrease with increasing β . In our stability analysis, this corresponds to a changes from an ill-posed to a well-posed but unstable gauge.

4.4 Instability of rotating reference frames

In one-dimensional cases considered above the cause of instability and ill-posedness was the perturbation of accelerations of test particles forming the reference frame due to perturbation of positions of these particles. In more than one dimensions, there is another physical factor, rotation and Coriolis forces associated with it. However, as we will see below, this factor does not change the nature of instability. Consider, for example, a uniformly rotating reference frame

$$\alpha = 1, \quad \beta_i = \{-\Omega x^2, \beta_2 = \Omega x^1, 0\}, \quad \gamma_{ij} = \text{diag}(1, 1, 1) . \quad (4.27)$$

where Ω is an angular velocity. For (4.27) we have

$$C^i = \Gamma_{00}^i + \beta^j \beta^k \Gamma_{jk}^i = \Gamma_{00}^i = \{\Omega^2 x^1, \Omega^2 x^2, 0\} , \quad (4.28)$$

and, according to our gauge stability analysis (3.23), a perturbation with a wave vector $q_i = q e_i$ will grow with the increment

$$\text{Re}(w) = \Omega q^{1/2} \sqrt{x^1 e_1 + x^2 e_2} . \quad (4.29)$$

Note, that C^i are proportional to components of physical acceleration a_i in (4.23). The increment $\propto q^{1/2}$ indicates that the gauge is ill-posed. The reason is the same as in a one-dimensional case. It is related to a radial acceleration and not to Coriolis force. Note that at the axis of rotation $x^1 = x^2 = 0$ we have $C^i = 0$ and there is no instability ($\text{Re}(\omega) = 0$ in (4.29)), but of course, there is a Coriolis force. Thus we conclude that in a three-dimensional case with rotation the main physical reason for gauge instability and ill-posedness is the same acceleration a^i .

5 Conclusions

In this paper we presented a general approach to the analysis of gauge stability of 3+1 formulations of GR. Gauge modes of perturbations can be separated, in a linear approximation,

from other modes of perturbations and studied independently. A system of eight quasi-linear partial differential equations (2.11), (2.13) describes the evolution of coordinate perturbations ψ_a and the corresponding perturbations of lapse and shift, U and V_k , with time. The gauge stability depend on the choice of gauge and on an unperturbed four-metric g_{ab} , but it does not depend on a particular form of a 3+1 system of GR equations.

Well-posedness and stability of several gauges was investigated. We demonstrated that all fixed gauges, i.e., gauges that are functions of coordinates only, are ill-posed with the exception of a synchronous gauge $\alpha = \alpha(t)$, $\beta_i = 0$. This gauge is well-posed, but it is prone to the formation of coordinate singularities (caustics) and it is unstable.

It is known that maximal slicing $tr(K_{ij}) = 0$ prevents the formation of coordinate singularities by applying accelerations to test particles forming a reference frame in such a way that $\frac{\partial \gamma}{\partial t} \propto tr(K_{ij}) = 0$ and the local volume element remain $\gamma^{1/2} = const.$ [37]. This, and a singularity-avoiding properties of maximal slicing allow in many cases a much longer numerical integration than using a synchronous gauge. However, both the maximal slicing gauge and its parabolic extension (3.42) are ill-posed due to the presence of acceleration-related unstable modes. Computations using these gauges will blow after a long enough period of integration.

Stability of metric-dependent algebraic gauges has been investigated as well. In particular, the necessary condition of well-posedness of gauges with metric-dependent lapse and fixed shift (3.26) was formulated. In addition to the formation of caustics, algebraic well-posed gauges with spatially dependent lapse and shift are susceptible to instabilities caused by perturbations of accelerations in deforming reference frames. The reason for the unstable behavior of these hyperbolic gauges is the same as that for ill-posedness of fixed, parabolic, or elliptic gauges. However, due to an “advective” property of hyperbolic gauges, the growth rate of acceleration-related unstable modes becomes wavelength-independent, $Re(\omega) \sim O(1)$, in the limit $q \rightarrow \infty$.

An investigation of stability of 3+1 formulations of GR in this paper is limited to an investigation of gauge instabilities. By studying ill-posedness and stability of (2.11), (2.13) one can tell if a 3+1 sets of GR equations using a particular gauge will be ill-posed or unstable. All gauges studied in this paper were found either ill-posed or unstable. None of the gauges investigated in this paper are suitable for a long-term stable integration of GR equations. Gauges with better stability properties must be found.

Stability of a gauge does not mean, however, that a 3+1 system using this gauge will be stable. As was mentioned in the introduction, another source of ill-posedness and instability is associated with violation of constraints. Constraint instabilities do depend on a particular form of a 3+1 system. The analysis of hyperbolicity in [19] provides an illustration of this statement. A part of the hyperbolicity conditions derived in this work (their equation (2.36)) does not involve gauge at all, and is dependent on how the constraint equations are incorporated into the system.

Acknowledgments

This work was supported in part by the NASA grant SPA-00-067, Danish Natural Science Research Council through grant No 94016535, Danmarks Grundforskningsfond through its support for establishment of the Theoretical Astrophysics Center, and by the Naval Research

Laboratory through the Office of Naval Research. The authors thank A. Doroshkevich, N. Khokhlova, M. Kiel, L. Lindblom, M. Scheel, K. Thorne, and M. Vishik for stimulating discussions and useful comments. I.D. thanks the Naval Research Laboratory, A.K. thanks the Theoretical Astrophysics Center, and both authors thank Caltech for hospitality during their visits.

Appendix: Physical meaning of A and B in (4.17)

Let us introduce the following new coordinates \tilde{t} , \tilde{x}^1 , \tilde{x}^2 , \tilde{x}^3 ,

$$\begin{aligned} T &= \frac{\tilde{t} + U\tilde{x}^1}{\sqrt{1-U^2}} , \\ X^1 &= \frac{\tilde{x}^1 + U\tilde{t}}{\sqrt{1-U^2}} + \int_0^{\tilde{x}^1} \left(\frac{\partial U}{\partial y} \right) \frac{y dy}{U\sqrt{1-U^2}} , \\ X^2 &= \tilde{x}^2 , \quad X^3 = \tilde{x}^3 , \end{aligned} \tag{A.1}$$

where T, X^i are Cartesian coordinates in a Minkowski spacetime $g_{ab} = \text{diag}(-1, 1, 1, 1)$. In these new coordinates we get the metric with the components

$$g_{00} = -1 , \quad g_{11} = (A + B\tilde{t})^2 , \quad g_{22} = g_{33} = 1 , \quad \text{others } g_{ik} = 0 , \tag{A.2}$$

where

$$A = 1 + \frac{\tilde{x}^1}{U(1-U^2)} \left(\frac{\partial U}{\partial \tilde{x}^1} \right) , \quad B = \frac{1}{(1-U^2)} \left(\frac{\partial U}{\partial \tilde{x}^1} \right) . \tag{A.3}$$

It is clear that (A.1) is a generalization of a Lorentz transformation with the velocity U depending on \tilde{x}^1 .

References

- [1] R. Arnowitt, S. Deser, and C.W. Misner. *Gravitation: An Introduction to Current research*, pages 227–265. Wiley, New York, 1962.
- [2] C. Bona and J. Masso. Einstein’s evolution equations as a system of balance laws. *Phys. Rev.*, D 40:1022–1026, 1989.
- [3] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and Jr. J. W. York. Einstein and Yang-Mills theories in hyperbolic form without gauge fixing. *Phys. Rev. Lett.*, 75:33773381, 1995.
- [4] S. Frittelli and O. A. Reula. First-order symmetric hyperbolic Einstein equations with arbitrary fixed gauge. *Phys. Rev. Lett.*, 76:4667–4670, 1996.
- [5] C. Bona and J. Masso. First order hyperbolic formalism for numerical relativity. *Phys. Rev.*, D 56:1022–1026, 1997.
- [6] C. Bona, J. Masso, E. Seidel, and J. Stela. First order hyperbolic formalism for numerical relativity. *Phys. Rev.*, D 56:3405–3415, 1997.
- [7] A. Anderson and Jr. J. W. York. Hamiltonian time evolution for general relativity. *Phys. Rev. Lett.*, 81:1154, 1998.
- [8] A. Arbona, C. Bona, J. Masso, and J. Stela. Robust evolution system for numerical relativity. *Phys. Rev.*, D60:104014, 1999.
- [9] M. Alcubierre, B. Brugmann, M. Miller, and Wai-Mo Suen. Conformal hyperbolic formulation of the Einstein equations. *Phys. Rev.*, D 60:064017, 1999.
- [10] T. W. Baumgarte and S. L. Shapiro. On the numerical integration of Einstein’s field equations. *Phys. Rev.*, D59:024007, 1999.
- [11] O. Brodbeck, S. Frittelli, P. Hubner, and O. A. Reula. The Cauchy problem and the initial boundary value problem in numerical relativity. *Journal of Math Physics*, 40:909, 1999.
- [12] P. Laguna. Linear-nonlinear formulation of Einstein equations for the two-body problem in general relativity. *Phys. Rev.*, D 60:084012, 1999.
- [13] M. A. Scheel, T. W. Baumgarte, G. B. Cook, and S. L. Shapiro. Treating instabilities in a hyperbolic formulation of Einstein’s equations. *Phys. Rev.*, D58:024007, 1999.
- [14] M. Shibata. Fully general relativistic simulation of merging binary clusters - spatial gauge condition. *Prog. Theor. Phys.*, 101:1199–1233, 1999.
- [15] M. Shibata. Fully general relativistic simulation of coalescing binary neutron stars: Preparatory tests. *Phys. Rev.*, D60:104052, 1999.

- [16] M. Alcubierre, B. Bruggmann, T. Dramlitsch, J. A. Font, P. Papadopoulos, E. Seidel, N. Stergioulas, and R. Takahashi. Towards a stable numerical evolution of strongly gravitating systems in general relativity: The conformal treatments. *Phys. Rev.*, D 62:044034, 2000.
- [17] Hisa-aki Shinkai and Gen Yoneda. Hyperbolic formulations and numerical relativity: experiments using Ashtekar’s connection variables. *Class. Quantum Grav.*, 17:4799–4822, 2000.
- [18] B. Kelly, P. Laguna, K. Lockitch, J. Pullin, E. Schnetter, D. Shoemaker, and M. Tiglo. Cure for unstable numerical evolutions of single black holes: Adjusting the standard ADM equations in the spherically symmetric case. *Phys. Rev.*, D 64:084013, 2001.
- [19] L.E. Kidder, M.A. Scheel, and S.A. Teukolsky. Extending the lifetime of 3D black hole computations with a new hyperbolic system of evolution equations. *Phys. Rev.*, D 64:064017, 2001.
- [20] P. Anninos, K. Camarda, J. Masso, E. Seidel, W. Suen, and J. Towns. Three-dimensional numerical relativity: The evolution of black holes. *Phys. Rev.*, D 52:2044–2058, 1995.
- [21] C. Bona, J. Masso, E. Seidel, and P. Walker. Three dimensional numerical relativity with a hyperbolic formulation. 1998. gr-qc/9804052.
- [22] E. Seidel. Numerical relativity: Towards simulations of 3D black hole coalescence. 1998. gr-qc/9806088.
- [23] E. Seidel and W.-M. Suen. Numerical relativity as a tool for computational astrophysics. *Journal of Computational and Applied Mathematics*, 109:493–525, 1999.
- [24] M. Alcubierre, W. Bengert, B. Brueggemann, G. Lanfermann, L. Nerger, E. Seidel, and R. Takahashi. The 3D grazing collision of two black holes. 2001. gr-qc/0012079.
- [25] M. Miller. On the numerical stability of the Einstein equations. 2000. gr-qc/0008017.
- [26] S. A. Teukolsky. Stability of the iterated Crank-Nicholson method in numerical relativity. *Phys. Rev.*, D61:087501, 2000.
- [27] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume II. John Wiley & Sons, New York, 1989.
- [28] W. E. Williams. *Partial Differential Equations*. Clarendon Press, Oxford, 1980.
- [29] A. Fischer and J. Marsden. *Comm. Math. Phys.*, 28:1, 1972.
- [30] R. M. Wald. *General relativity*. The University of Chicago Press, Chicago, 1984.
- [31] L. D. Landau and E. M. Lifshits. *The Classical Theory of Fields*. Butterworth-Heinemann, Oxford, 1975.
- [32] E.M. Lifshitz, V.V. Sudakov, and I.M. Khalatnikov. *Soviet Physics JETP*, 13:1298, 1961.

- [33] M. Alcubierre, G. Allen, and B. Bruggmann. Towards an understanding of the stability properties of the 3+1 evolution equations. *Phys. Rev.*, D 62:124011, 2000.
- [34] M. Alcubierre and J. Masso. Pathologies of hyperbolic gauges in general relativity and other field theories. *Phys. Rev.*, D 57:R4511, 1998.
- [35] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman and Company, New York, 1973.
- [36] L. Smarr and W. York. Radiation gauge in general relativity. *Phys. Rev.*, D 17:1945–1956, 1977.
- [37] L. Smarr and W. York. Kinematical conditions in the construction of spacetime. *Phys. Rev.*, D 17:2529–2551, 1977.
- [38] W. Rindler. *Am. J. of Phys.*, 34:1174, 1966.
- [39] V. P. Frolov and I. D. Novikov. *Black Hole Physics*. Kluwer Academic Publishers, Dordrecht, 1998.
- [40] A. L. Zelmanov. *Docl. Acad. Nauk USSR*, 107:815, 1956.